# When the Recursive Diversity Anonymity Meets the Ring Signature 

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#### Abstract

In privacy-preserving blockchain systems, to protect a sender's identity of a transaction in privacy-preserving blockchain systems, ring signature (RS) schemes have been widely implemented, which allow users to obscure consumed tokens via including "mixin" (i.e., chaff tokens). However, recent works point out that existing RS schemes are vulnerable to the "chain-reaction" analysis, where adversaries eliminate mixins of RSs by utilizing the fact that each token can only be consumed in a RS. By "chain-reaction" analysis, adversaries can find some definite token-RS pair sets (DTRSs) to confirm the sender's identity of a RS. Besides, the existing RS schemes do not consider the diversity of mixins when generating a RS. Moreover, since the transaction fee is proportional to the number of mixins, a use is motivated to use a RS with the minimum number of mixins. In this paper, we formally define the diversity-aware mixins selection (DA-MS) problem, which aims to generate a RS with the minimum number of mixins satisfying the constraints of its diversity and the anonymity of other RSs. We prove the DA-MS problem is \# $P$ and propose a breadth-first search algorithm to get the optimal solution. Furthermore, to efficiently solve the DA-MS problem, we propose two practical configurations and two approximation algorithms with theoretic guarantees. Through comprehensive experiments on real data sets as well as synthetic data sets, we illustrate the effectiveness and the efficiency of our solutions.


## CCS CONCEPTS

- Security and privacy $\rightarrow$ Privacy protections.


## KEYWORDS

Blockchain, Ring Signature, Privacy
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Figure 1: The Unspent Transaction Output (UTXO) model.
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## 1 INTRODUCTION

Recently, blockchain technologies attract much attention from both academia and industries. From the perspective of the database community, blockchain is a kind of the distributed transaction management solution, where nodes keep replicas data of publicly agreed execution order of the transactions [1]. Many blockchain applications, such as cryptocurrency [2], e-voting [3], healthcare [4], and storage sharing [5], are in the Unspent Transaction Output (UTXO) model, where each transaction will consume some input token(s) from its sender(s) to its receiver(s). Thus, one token can only be consumed once. In blockchain systems, tokens represent programmable assets or access rights, which are operable only by the persons who have their private keys. For example, in e-voting systems [3], a token indicates a ballot of a user.

Since the transaction data is available for all the nodes in the blockchain system, the privacy issue (e.g., the ownership of tokens) needs to be handled for specific applications (e.g., e-voting or healthcare systems). By analyzing transactions on blockchains, adversaries can link users to transactions/tokens [6-8]. For instance, if a transaction $h_{2}$ consumes a token which is generated in a historical transaction (HT) $h_{1}$, adversaries can link the receiver of $h_{1}$ to the sender of $h_{2}$. Although current blockchain systems conceal users' identities by pseudonymities, this kind of link is still risk. Once adversaries link one pseudonymity with a user's identity, they can link the user's identity to all pseudonymities that are linked to the revealed pseudonymity [9]. Furthermore, they can infer who are trade partners with this user.

To protect privacy, ring signature (RS) schemes are widely used in various blockchain systems, where users can utilize RSs to obscure the ownership of input tokens by adding chaff tokens, namely "mixins". In some scenarios, users use RSs to hide the consumed
tokens in transactions to avoid adversaries linking their identities to transactions/tokens. As illustrated in Figure 1, there are multiple input RSs and output tokens in a transaction. A RS contains only one consumed token marked in yellow color and many other tokens (mixins) marked in green color. The details of RS schemes will be introduced in Section 2. Currently, existing RS schemes measure the anonymity of a RS as its number of mixins [10, 11]. However, there are two defects of these existing methods. The first is that, if in a RS the consumed token and its mixins are generated by the same transaction $h_{i}$, the source of the consumed token can still be inferred as $h_{i}$, which is called Homogeneity Attack [12]. The second is that, since each token can only be consumed once, adversaries can infer the consumed token in a RS by analyzing other related RSs, which is called Chain-reaction Analysis [10]. Besides, since the transaction fee is proportional to the number of mixins, a user is motivated to find an anti-attack RS with the smallest size.

To overcome the weakness of the existing solutions, in this paper, we will consider a RS generation problem for blockchain systems, namely diversity-aware mixins selection (DA-MS), which selects a minimal set of mixins for a given token to generate a valid antiattach RS. Here is a motivation example.

Example 1 (An Mixins Selection Example). Suppose there are four tokens $\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$ and two RSs, $r_{1}=\left\{t_{1}, t_{2}\right\}$ and $r_{2}=\left\{t_{1}, t_{2}\right\}$. The tokens $t_{1}$ and $t_{3}$ are the outputs of the same HT $h_{1}$. The tokens $t_{2}$ and $t_{4}$ are outputs of another two $H T s h_{2}$ and $h_{3}$, respectively. Now we want to generate a RS for consuming $t_{3}$.

The first solution is using $r_{3}=\left\{t_{1}, t_{3}\right\}$. Since $t_{1}$ and $t_{3}$ are all generated in $h_{1}$, adversaries need not to determine the consumed token in $r_{3}$ and can directly know the consumed token of $r_{3}$ is from $h_{1}$, which does not resist the homogeneity attack.

The second solution is using $r_{3}=\left\{t_{2}, t_{3}\right\}$. Recall that each token can only be consumed once. Since $r_{1}=r_{2}=\left\{t_{1}, t_{2}\right\}$, adversaries can conclude that $t_{1}$ and $t_{2}$ must have been consumed. Thus, the consumed token in $r_{3}=\left\{t_{2}, t_{3}\right\}$ must be $t_{3}$, which means chain-reaction analysis is not resisted in this solution.

The third solution is using $r_{3}=\left\{t_{1}, t_{2}, t_{3}, t_{4}\right\}$. The consumed tokens of $r_{1}, r_{2}$, and $r_{3}$ cannot be inferred. However, $r_{3}$ 's size is 4.

A good solution is using $r_{3}=\left\{t_{3}, t_{4}\right\}$. The consumed tokens in $r_{1}$, $r_{2}$, and $r_{3}$ cannot be inferred. Besides, this $r_{3}$ contains only 2 tokens.

Motivated by the above example, we will formalize the DA-MS problem, which targets on selecting a minimal set of mixins for a given token to consist a RS such that the new RS and existing RSs can resist attacks (e.g., homogeneity attacks and chain-reaction analysis attacks). In this paper, we first define a concept called "definite token-RS pair set" (DTRS) to resist the attack based on the chain-reaction analysis. Besides, inspired by the recursive diversity principle [13], we define the concept of a "recursive diversity RS" to measure the anonymity of a RS. Then, based on these concepts, we formally define the DA-MS problem as to find a recursive diversity RS with minimal size while retaining the existing RSs' privacy. We prove the DA-MS problem is \#P through a reduction from the problem of enumerating all perfect matchings in bipartite graphs [14]. We first propose an exact breadth-first search algorithm. However, the time complexity is exponential high. To efficiently tackle DAMS, three challenges need to be addressed: (a) how to efficiently verify if DTRSs of a RS satisfy the diversity requirement; (b) how to guarantee that a user can always consume a token with a RS
that retains existing RSs' diversities since the blockchain system is distributed; and (c) how to pick an eligible set of mixins with a minimal size. To overcome the first two challenge, we propose two practical configurations which require that every new RS must be the superset of an existing RS or be disjoint with it; and the diversity requirement of a new RS should be a little higher than the diversity requirement of its DTRSs. To tackle the third challenge, we propose two approximation algorithms, namely the Progressive Algorithm and the Game-theoretic Algorithm, with theoretic guarantees.

Specifically, we make the following contributions:

- We define the threat model and the notion of a recursive ( $\epsilon$, l)-diversity RS in Section 2.
- We formally define the diversity-aware mixins selection (DA-MS) problem and give the proof of its hardness in Section 3.
- We propose a breadth-first search algorithm to achieve the exact solution of DA-MS in Section 5.
- We propose two practical configurations and two approximation algorithms, the Progressive Algorithm and the Game-theoretic Algorithm, with theoretic guarantees in Section 6.
- We conduct comprehensive experiments on real and synthetic data sets to demonstrate the effectiveness and efficiency of our solutions in Section 7.
Besides, we propose a framework, TokenMagic, in Section 4, discuss the related work in Section 8, and conclude in Section 9.


## 2 PRIVACY SEMANTICS

In blockchain systems, since RSs can be verified by all users (e.g., miners and transaction users), adversaries can access all the data of historical RSs. Thus, different from many privacy scenarios where users preserve privacy through bringing in noises [6-8], users must guarantee that RSs can be accurately executed in blockchain systems. Since users in blockchain systems are usually anonymous to some extent (i.e., using different unknown public keys), adversaries want to link some transactions on blockchains together and infer more information about transactions [9] (e.g., the addresses of partners of trades). Moreover, blockchain systems are distributed. Each user customizes RSs for her/his own benefits (i.e., minimizing the sizes of her/his own RSs) and privacy requirements. In this section, we introduce the threat model in blockchain scenarios and define the notion of a recursive $(c, \ell)$-diversity RS .

### 2.1 The Ring Signature Scheme

Generally, a RS scheme can be divided into 3 parts: selecting a set of mixins, generating a ring signature, and verifying the signature:

- Step 1. Selecting a set of mixins $M$. For example, in Monero [2], a user inputs an integer $\zeta$ (larger than 10) to algorithm $S M$, then $S M$ retrieves half of $\zeta$ mixins from the blocks generated within recent 1.8 days and other mixins from other blocks.
- Step 2. Generating a ring signature. With transaction message $m g$, a set of mixins $M$, the consuming token $t$ and its private key $p k$, the algorithm Gen outputs a RS $r$, which contains the set of mixins $M$, a token image $I$, and a set of auxiliary parameters $\omega$.
- Step 3. Verifying the signature. When other users receive $r$, they verify if $r$ is eligible. If $I$ had already been used (indicating the corresponding token having been consumed), $r$ will be rejected. With $\omega$, the algorithm Ver can verify if the user has the private key of $t$. Besides, verifiers can check if $r$ satisfies some extra
configurations (e.g., Monero requires half of mixins must come from blocks generated in recent 1.8 days). If $r$ conflicts these configurations, $r$ will also be rejected.
Step 1 and Step 2 are conducted offline by users themselves. Step 3 is verified by miners when they block transactions. Thus, only the time complexity of Step 3 has an impact on the throughput of the blockchain system. A RS is a combination of a sorted sequence of public keys of tokens $(M \cup t)$, a token image $I$ of $t$, and a zeroknowledge proof $\omega$. For a token $t$, its owner can use its private key $p k$ to generate one token image $I$, which is explicit to all users. Observers cannot infer $t$ from I. For a token, its image is unique. When an image $I$ was used, we know the corresponding token was used and cannot be used again (i.e., prevent the double-spending attack [15]), although we do not know which token it is. The zeroknowledge proof [16] is used to prove that the user knows the index of the corresponding token of the image in the sorted sequence, while the proof will not leak any information about the token or its index. A user can use a RS to prove she/he has the right to use an unconsumed token and do not let others know what the token is. In this paper, we focus on how to select a desired set of mixins in Step 1 to enhance anonymity. This paper does not involve any change of the encryption algorithm in Step 2 and the verification algorithm in Step 3. In the rest of this paper, we simply consider a RS as a set of tokens consisting of a consuming token and its mixins.


### 2.2 The related RS set of a RS

Definition 1 (The Related RS Set of a RS). For a RS $r_{k}$, at timestamp $\pi$, the related RS set of $r_{k}$ is $R_{\pi}^{r_{k}}=\bigcup_{i=0} R_{\pi}^{r_{k}, i}$ where $R_{\pi}^{r_{k}, i}$ is the set of RSs which are proposed before timestamp $\pi$ and contain some tokens in common with some $\operatorname{RS}(\mathrm{s})$ in $R_{\pi}^{r_{k}, i-1}$.

When $i=0, R_{\pi}^{r_{k}, 0}$ is the set of RSs containing any tokens in $r_{k}$.
Example 2. At time stamp $\pi$, there are four $R S s, r_{1}=\left\{t_{1}, t_{2}, t_{5}\right\}$, $r_{2}=\left\{t_{1}, t_{3}\right\}, r_{3}=\left\{t_{1}, t_{3}\right\}, r_{4}=\left\{t_{2}, t_{4}\right\}$, and $r_{5}=\left\{t_{4}, t_{5}, t_{6}\right\}$. The tokens $t_{5}$ and $t_{6}$ are from the same historical transaction $h_{1}$.

By Definition 1, in Example 2, $R_{\pi}^{r_{4}}=\left\{r_{1}, r_{2}, r_{3}, r_{5}\right\}$. Specifically, $R_{\pi}^{r_{4}, 0}=\left\{r_{1}, r_{5}\right\}$ and $R_{\pi}^{r_{4}, 1}=\left\{r_{2}, r_{3}\right\}$. Thus, in some cases, the related RS set would be very large. In Section 4, we propose a framework to limit the size of the related RS set for a given RS.

### 2.3 The definite token-RS pair set (DTRS)

Definition 2 (The Definite Token-RS Pair Set). At timestamp $\pi$, given a RS $r_{k}$, a definite token-RS pair set (DTRS) of $r_{k}$ is a minimum set of token-RS pairs $d^{\pi, k}=\left\{p_{1}, p_{2}, \cdots, p_{n}\right\}$ that can determine the HT of the consumed token of $r_{k}$, where $p_{i}=\left\langle t_{i}, r_{i}\right\rangle$ is a token-RS pair indicating the token $t_{i}$ is consumed in the RS $r_{i}$.

Some researchers have proposed methods (e.g., chain-reaction analysis) to cluster users through their historical transactions on the blockchain [17-19]. In chain-reaction analysis, the main purpose of adversaries is to infer the link between a RS and an entity [19]. For a given RS $r_{k}$ in timestamp $\pi$, if the token-pairs of its DTRS $d^{\pi, k}$ are revealed to adversaries, they can exactly determine the consumed token in $r_{k}$. In Example 2, $\left\{\left\langle t_{2}, r_{1}\right\rangle\right\}$ is a DTRS of $r_{5}$. Because when $t_{2}$ is consumed in $r_{1}$, the consumed token in $r_{4}$ must be $t_{4}$. Thus, the consumed token in $r_{5}$ must be $t_{5}$ or $t_{6}$, who are from HT $h_{1}$.

### 2.4 The Threat Model

Since blockchain systems are distributed, users can use different random strategies. Besides, since a token can only be consumed
once, the procedure of generating a RS for a token happens only once. Thus, adversaries cannot guess the probability distribution of consumed tokens. However, because all data of RSs are recorded on the blockchain and a token can only be consumed in a RS, adversaries can guess the consumed token for a RS by analyzing the transactions on the blockchain (i.e., chain-reaction analysis) [10].

As common adversary assumptions [20], we assume adversaries know the algorithms used in blockchain systems, except for random procedures of algorithms. Thus, adversaries cannot exactly infer the real consumed token for a given RS without using any other information. Besides, we assume adversaries have some side information, i.e., they may know some token-RS pairs are revealed.

Definition 3 (The side information of adversaries). Before proposing a new RS $r_{i}$, the side information of adversaries is a set of revealed token-RS pairs, $S I_{i}=\left\{p_{1}, p_{2}, \cdots, p_{n}\right\}$. A pair $p_{j}=\left\langle t_{j}, r_{j}\right\rangle \in$ $S I_{i}$ means that adversaries know the consumed token of $r_{j}$ is $t_{j}$.

The side information of adversaries can be partitioned into two disjoint parts $S I_{i}^{\#}$ and $S I_{i}^{*}$ (i.e., $S I_{i}=S I_{i}^{\#} \cup S I_{i}^{*}$ and $S I_{i}^{\#} \cap S I_{i}^{*}=\emptyset$ ), where $S I_{i}^{\#}$ is the set of token-RS pairs that adversaries know directly, and $S I_{i}^{*}$ is the set of token-RS pairs that adversaries infer from $S I_{i}^{\#}$. For example, adversaries may be the generator of some RSs, then their token-RS pairs $S I_{i}^{\#}$ are known to themselves [11].

Adversaries have two methods to get $S I_{i}^{\#}$. Firstly, they can use side information to eliminate tokens in the target RS and infer possible token-RS pairs by the frequency of HTs of remaining tokens (similar to the homogeneity attacks [13]). In Example 1, if adversaries know $t_{2}$ and $t_{4}$ are not consumed in $r_{3}=\left\{t_{1}, t_{2}, t_{3}, t_{4}\right\}$, they can infer that the generator of $r_{3}$ is the receiver of the HT $h_{1}$ [9], since $t_{1}$ and $t_{3}$ are the outputs of $h_{1}$. The second method is to eliminate wrong DTRSs. In Example 2, there are three DTRSs of $r_{4}, d_{1}^{\pi, 4}=\left\{\left\langle t_{4}, r_{5}\right\rangle\right\}, d_{2}^{\pi, 4}=\left\{\left\langle t_{5}, r_{5}\right\rangle\right\}$, and $d_{3}^{\pi, 4}=\left\{\left\langle t_{2}, r_{1}\right\rangle\right\}$. If adversaries know $t_{5}$ is consumed in $r_{5}$, they can eliminate $d_{1}^{\pi, 4}$ and $d_{3}^{\pi, 4}$, and conclude that $t_{4}$ is the consumed token of $r_{4}$. Thus, the more tokens of a RS and its possible DTRSs are from different HTs, the better anonymity of a RS would be. This provides a natural motivation to apply the recursive diversity technique to enhance anonymity. However, if the side information of adversaries is strong enough (e.g., they know all consumed tokens of existing RSs), we cannot protect users' privacy. In Theorem 6.2, we prove that if the cardinality of $S I$ is smaller than a threshold, with TokenMagic framework, adversaries cannot confirm the consumed tokens.

### 2.5 A Recursive ( $c, \ell$ )-Diversity RS

We borrow the principle of recursive ( $c, \ell$ )-diversity [13] to define the notion of a recursive $(c, \ell)$-diversity RS. The recursive $(c, \ell)$ diversity requires that $q_{1}<c \cdot\left(q_{\ell}+\cdots+q_{\theta}\right)$, where $q_{i}$ is the number of times that the $i^{t h}$ most-frequent sensitive value appears in the data set, and $\theta$ is the number of possible sensitive values in the data set. In RS scenarios, the sensitive value is the HT of a token. Parameter $c$ and $\ell$ are set by users according to their requirements. Definition 4 (A Recursive ( $c, \ell$ )-Diversity RS). At timestamp $\pi$, a RS $r_{k}$ is a recursive ( $c, \ell$ )-diversity RS, if (1) the HTs of tokens in $r_{k}$ satisfy the recursive ( $c, \ell$ )-diversity; and (2) the HTs of tokens in any DTRS of $r_{k}$ satisfy the recursive ( $c, \ell$ )-diversity.

Suppose at timestamp $\pi$, there are three RSs, $r_{1}=\left\{t_{1}, t_{2}\right\}, r_{2}=$ $\left\{t_{2}, t_{3}\right\}$, and $r_{3}=\left\{t_{1}, t_{3}, t_{4}\right\}$. The tokens $t_{1}$ and $t_{3}$ are from the same

Table 1: Symbols and Descriptions.

| Symbol | Description |
| :--- | :--- |
| $T$ | The universe of tokens $t_{i}$ |
| $h_{i}$ | The HT that outputs the token $t_{i}$ |
| $R_{\pi}^{r_{k}}$ | The related RS set of a RS $r_{k}$ at the time stamp $\pi$ |
| $\left(c_{k}, \ell_{k}\right)$ | The diversity requirement of a RS $r_{k}$ |
| $p_{k}$ | A token-RS pair $\left\langle t_{k}, r_{k}\right\rangle$ that means $t p_{k}$ is consumed in $r p_{k}$ |
| $d^{\pi, k}$ | A DTRS of a RS $r_{k}$ at the time stamp $\pi$ |

HT $h_{1}$, and $t_{4}$ is from $h_{2}$. Thus, the HTs of tokens in $r_{3}$ are $h_{1}, h_{1}$ and $h_{2}$, respectively. Thus, for the HTs of tokens in $r_{3}, q_{1}=2$ and $q_{2}=1$. For $r_{3}$, there is only one DTRS, $d^{\pi, 3}=\left\{\left\langle t_{1}, r_{1}\right\rangle,\left\langle t_{3}, r_{2}\right\rangle\right\}$. If a user requires a recursive (2,1)-diversity RS, $r_{3}$ is eligible for both two conditions, i.e., $2<2 \cdot(2+1)$ and $2<2 \cdot 2$. If a user requires a recursive (3, 2)-diversity RS, $r_{3}$ satisfies the first condition, i.e., $2<3 \cdot 1$, but violates the second condition, i.e., $2 \geq 3 \cdot 0$.

## 3 PROBLEM DEFINITION

In this section, we formally formulate the Diversity-aware mixins selection (DA-MS) problem and prove that DA-MS is \#P.

### 3.1 Diversity-aware Mixins Selection Problem

After a RS is blocked on the blockchain, its DTRSs and its anonymity may still be changed. For example, in Example 2, if a new RS $r_{6}=$ $\left\{t_{2}, t_{4}\right\}$ is proposed, adversaries can infer that the consumed token of $r_{1}$ is $t_{5}$ and the consumed token of $r_{5}$ is $t_{6}$. To maintain privacy, a user can claim the anonymity requirement when committing a RS to the blockchain. When a new RS is proposed, it should guarantee that the DTRSs of each existing RS still satisfy its claimed anonymity requirement. We define the DA-MS problem as follows:
Definition 5 (The Diversity-aware Mixins Selection (DA-MS) Problem). Given a mixin universe $T$, a token $t_{\tau}$ that will be consumed at timestamp $\pi$, and a privacy requirement $\left(c_{\tau}, \ell_{\tau}\right)$, a user wants to select a mixin set, to generate a new $\operatorname{RS} r_{\tau}$, such that its cardinality is minimized and the following constraints are satisfied:

- Diversity constraint: $r_{\tau}$ is a recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity RS;
- Non-eliminated constraint: no token of any RS can be eliminated by the "Chain-Reaction" analysis;
- Immutability constraint: after proposing $r_{\tau}$, each RS $r_{i}$ in $R_{\pi}^{r_{\tau}}$ can maintain its requirement of recursive ( $c_{i}, \ell_{i}$ )-diversity.
To guarantee each RS has desired anonymity, the system will require that each new RS must not violate the recursive diversity requirements of the existing RSs.


### 3.2 Hardness of the DA-MS Problem

We prove that the DA-MS problem is \#P by reducing the problem of enumerating all perfect matchings in bipartite graphs [14].

Theorem 3.1. The DA-MS problem is \#P.
Proof. We first prove the decision version of the DA-MS problem (DDA-MS) is \#P. Given a new RS $r_{\tau}$, a set of related RS set $R_{\pi}^{r_{\tau}}$, and a privacy requirement $\left(c_{\tau}, \ell_{\tau}\right)$, the DDA-MS aims to prove if $r_{\tau}$ satisfies the tree constraints in Definition 5.

We prove DDA-MS is \#P by a reduction from the problem of enumerating all perfect matchings in bipartite graphs (EPMBG) [14]. Let $G=\left(S_{1}, S_{2}, E\right)$ be a bipartite graph with row vertex set $S_{1}$, column vertex set $S_{2}$, satisfying that $\left|S_{1}\right|=\left|S_{2}\right|=n$, and edge set $E \subseteq S_{1} \times S_{2}$. A matching $M$ is a subset of $E$, where any two edges do not intersect at the same vertex. If each vertex in $G$ is covered


Figure 2: TokenMagic.
by an edge in $M, M$ is termed as a perfect matching. The EPMBG problem aims to enumerate all perfect matchings in the $G$.

Given an EPMBG problem instance, we reduce it to a DDAMS problem instance as follows: Generate a related RS set $R_{\pi}^{T}=$ $\left\{r_{1}, \cdots, r_{n-1}\right\}$, where $r_{i}$ is the set of vertexes in $S_{2}$ which are connected with the $i^{\text {th }}$ vertex in $S_{1}$. The new RS $r_{\tau}$ is the set of vertexes in $S_{2}$ which are connected with the $n^{t h}$ vertex in $S_{1}$. To verify if $r_{\tau}$ is a recursive ( $c, \ell$ )-diversity RS, we need to enumerate all possible token-RS pairs of RSs in $R_{\pi}^{T} \cup r_{\tau}$. Since each token can only be consumed once, the set of perfect matchings in $G$ is the set of token-RS combinations when the consumed token in $r_{\tau}$ is $t_{\tau}$.

Thus, if the transformed DDA-MS problem instance can be solved, the EPMBG problem instance also can be solved. Since the EPMBG problem is \#P [14], the DDA-MS problem is also an \#P problem. Since the DA-MS problem is an optimization problem, the hardness of the DA-MS problem is at least \#P.

## 4 TOKENMAGIC FRAMEWORK

Since a new RS's anonymity is impacted by the related RSs, before selecting mixins, a user needs to retrieve the mixin universe $T$ and the related RS set $A_{\pi}^{r_{\tau}}$. The mixin universe can be retrieved by the user's interest or the blockchain system's requirement (e.g., Monero [2] requires half of the mixins are generated within the recent 1.8 days). As discussed in Subsection 2.2, by this method, the size of a related RS set may be very large and can unlimitedly increase over time. Take an extreme example, if any two existing RSs contain at least one same token, the related RS set would contain all RSs on the blockchain. To generate a new RS, a user needs to retrieve all RSs and find DTRSs among these RSs, which is costly.

In this paper, we propose a novel framework, namely TokenMagic, to retrieve the mixin universe $T$ as well as the related RS set $R_{\pi}^{r_{\tau}}$ and generate a new $\operatorname{RS} r_{\tau}$ for a given token $t_{\tau}$ at timestamp $\pi$. As shown in Figure 2, TokenMagic partitions the blocks in a blockchain into disjoint and sequential batches $\mathbb{B}=\left[B_{1}, B_{2}, \cdots\right]$. The number of tokens in each batch $B_{i}$ is bounded in a range $\lambda$ specified by the system. Specifically, when we build the batch list $\mathbb{B}$, we scan from the first block $b_{1}$ in the blockchain system and initialize the first batch $B_{1}$ as empty. We traverse the blocks in ascending order. Every time we visit a block $b_{x}$, we check whether the current batch $B_{i}$ has enough tokens after adding $b_{x}$ (i.e., $t\left(b_{x}\right)+\sum_{b_{j} \in B_{i}} t\left(b_{j}\right) \geq \lambda$, where $t\left(b_{i}\right)$ indicates the number of tokens in $b_{i}$ ). If $B_{i}$ has enough tokens, we move to build the next batch $B_{i+1}$. For batch $B_{i}$, let $T_{i}$ be the tokens in the blocks in $B_{i}$.

For full-node users [21] who store full blockchain data in their devices, they can build the batch list by themselves and update the batch list when they update their blockchain state from the local

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Algorithm 1: TokenMagic Framework.
    Input: the consuming token \(t_{\tau}\), and the required diversity constraint ( \(c\),
            e)
    Output: an new RS to consume \(t_{\tau}\)
    retrieve the tokens \(T\) of the batch where \(t_{\tau}\) is generated;
    initialize Cand \(_{i}=\emptyset\) for each \(t_{i} \in T\);
    foreach token \(t_{i}\) in \(T\) do
        use BFS, Progressive, or Game-theoretic approaches to generate a
        new RS \(r\) satisfying the constraints in Definition 5;
        foreach token \(t_{i}\) in \(r\) do
            add \(r\) to Cand \(_{i}\);
    return a RS from Cand \(_{\tau}\) randomly;
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state to the global state. Since $\lambda$ is a public system parameter and users have a consensus about the block list in the blockchain system, users can have a consensus about the batch list too. For light-node users [21] who do not store full blockchain data in their devices, they can query the wanted batch data from full-node users. For each consuming token, its mixins are only selected within the same batch where the consuming token is generated. Note that there is a gap between the timestamp when a token is generated and the timestamp when the token is consumed by a RS in a transaction. For example, in Figure 2, a token is generated in the block $b_{1}$, but the RS which contains $b_{1}$ is proposed in the block $b_{g+1}$.

Thus, to generate a RS for a token $t$, the mixin universe is the set of tokens that are generated in the same batch of $t$. Thus, cardinality of the mixin universe is limited to the number of tokens in the corresponding batch. Since batches are disjoint, the mixin universes of tokens in different batches are disjoint. Furthermore, since each RS selects mixins only from its mixin universe, the related RS sets of RSs in different batches are also disjoint. Because each token can only be consumed once, the cardinality of the related RS set of a RS is bounded by the cardinality of the mixin universe. Thus, the cardinality of the related RS set is bounded by the number of tokens in the corresponding batch.

Algorithm 1 is the pseudocode of the TokenMagic framework. We first retrieve the batch $b$ where $t_{\tau}$ is (line 1). To avoid adversaries inferring the consumed token by the framework and algorithms, we generate a new RS with a random procedure (line 2-5). Specifically, we first initialize candidate set Cand $_{i}$ of each token $t_{i}$ as empty (line 1). Then, for each token $t_{i}$, we use our BFS, Progressive, or Game-theoretic approach to generate a new RS $r$ satisfying the constraints of diversity, immutability, and non-eliminated (line 3). Since the new RS $r$ may be eligible for multiple tokens, we add $r$ to candidate sets of these tokens (line 4-5). After that, we randomly select a RS from Cand $_{\tau}$, where Cand $_{\tau}$ is the candidate RS set of the input token $t_{\tau}$ (line 6). Since blockchain systems are distributed, and the random selection in line 6 is made by users themselves, adversaries cannot observe or guess the probability distribution of selection. Therefore, adversaries cannot further infer the consumed token of a RS by the framework and algorithms.

Since the diversity requirement is customized by the user, if users think the returned RS is not desirable (e.g., the size is too large) or the framework cannot return an eligible RS, they can relax the diversity requirement by increasing $c$ or decreasing $\ell$. Besides, since a RS only selects mixins within the same batch where the
consumed token is, there could be a situation that a user cannot find a RS satisfying constraint of non-eliminated. Although we cannot confirm the consumed token of each RS, we still can find out that some tokens have been consumed. In Example 1, suppose the user chooses $r_{3}=\left\{t_{1}, t_{3}\right\}$ and the mixin universe is $T=\left\{t_{1}, t_{2}, t_{3}, t_{4}\right\}$. When another user wants to consume $t_{4}$, she/he cannot find a RS $r$ satisfying non-eliminated constraint, since adversaries can easily find that $t_{1}, t_{2}$, and $t_{3}$ have been consumed in the previous three RSs (i.e., $r_{1}, r_{2}, r_{3}$ ) and $t_{4}$ must be the consumed token in the new RS $r$. To avoid this issue, we require that when there are $i$ RSs in the possible related RS set $R_{\pi}^{T}$, the number of infer-able consumed tokens cannot exceed $i-\eta \cdot(|T|-i)$, where $\eta$ is a system parameter.

Theorem 4.1. Given a set of RSs, $R^{*}=\left\{r_{1}, \cdots, r_{n}\right\}$, suppose $T^{*}=$ $\bigcup_{i=1}^{n} r_{i} . I f\left|T^{*}\right|=n$, the tokens in $T^{*}$ are all consumed.

Proof. If a token in $T^{*}$ has not been consumed, there are only $n-1$ tokens have been consumed in $R^{*}$. By the Pigeonhole principle [22], there is at least a token that has been consumed in two RSs, which is impossible. Thus, the proof is completed.

Let $n s_{j}=<r_{1}, \cdots, r_{m}>$ be a "neighbor" set of token $t_{j}$ where $r_{i}$ contains $t_{j}$ and RSs are sorted by the proposed timestamps. Let $T_{j}^{\#}=$ $\bigcup_{r_{i} \in n s_{j}} r_{i}$. By Theorem 4.1, if the number of tokens in a neighbor set is equal to the number of RSs in the neighbor set (i.e., $\left.\left|T_{j}^{\#}\right|=\left|n s_{j}\right|\right)$, the corresponding token has been consumed. We store a neighbor set for each token and append RSs which contain the token to the neighbor set. Suppose, when the number of RSs that consumes token in $T$ is $i$, the number of tokens that can be determined to have been consumed is $\mu_{i}$. To avoid that users cannot find eligible RSs to consumed their tokens, we require that at any moment $i-\mu_{i} \geq \eta \cdot(|T|-i)$. When the number of tokens in a batch is smaller than $\lambda$, we consider $|T|$ as $\lambda+\lambda^{\prime}-1$.
The overhead of TokenMagic framework. The TokenMagic is implemented for the Step 1 of a RS scheme introduced in Section 2.1. The Step 1 is made offline and its time complexity does not affect the throughput of a blockchain system. TokenMagic brings extra overhead to the Step 1 of RS scheme, but no extra overhead to the throughput of the online blockchain system. In conclusion, our framework only affects the speed to offline generate new RSs, but does not affect the online blockchain processes at all.

## 5 AN EXACT APPROACH FOR DA-MS

As illustrated in Subsection 3.2, the DA-MS problem is \#P. However, if the size of the input (i.e, $|T|$ ) is small enough, brute-force methods still can find an optimal solution. In this section, we propose a breadth-first search (BFS) algorithm, to illustrate the problem space of DA-MS. We first introduce a basic concept as follows:
Definition 6 (Token-RS Combination). Given a RS set $R$, a tokenRS combination of $R$ is a set of $|R|$ token-RS pairs, denoted by $u=\left\{p_{1}, p_{2}, \cdots, p_{|R|}\right\}$ where $\forall p_{i}, p_{j} \in u, t_{i} \neq t_{j}$ and $r_{i} \neq r_{j}$.

In Example 1, when the new RS is $r_{3}=\left\{t_{2}, t_{3}, t_{4}\right\},\left\{\left\langle t_{1}, r_{1}\right\rangle\right.$, $\left.\left\langle t_{2}, r_{2}\right\rangle,\left\langle t_{3}, r_{3}\right\rangle\right\}$ is a token-RS combination of the RSs.

### 5.1 The BFS Approach

Algorithm 2 illustrates the BFS approach, which applies the breadth first search strategy to check all possible valid RSs and returns the

```
Algorithm 2: BFS Approach.
    Input: the consuming token \(t_{\tau}\), a mixin universe \(T\), and the required
            diversity constraint \((c, \ell)\)
    Output: an eligobile RS to consume \(t_{\tau}\)
    \(\sigma=T \backslash t_{\tau} ;\)
    for \(i=l_{\tau}-1\) to \(|\sigma|\) do
        \(C R^{i} \leftarrow\) possible RSs with \(i\) tokens from \(\sigma\);
        for \(c r \in C R^{i}\) do
            \(r s=t_{\tau} \cup c r\), retrieve \(R_{\pi}^{r s} ;\)
            \(H \leftarrow\) historical transactions of tokens in \(r s\);
            if \(H\) does not satisfy the recursive \((c, \ell)\)-diversity then
                continue;
            \(U \leftarrow\) all token-RS combinations of \(r s \cup R_{\pi}^{r s} ; k=0\);
            while \(k \leq\left|R_{\pi}^{r s}\right|\) do
                \(S T=\emptyset, k++;\)
                foreach \(u_{j}\) in \(U\) do
                    if \(w_{j, k} \notin S T\) then
                        \(S T=S T \cup w_{j, k} ;\)
                if \(S T \neq r_{k}\) then
                go to Line 4 for the next for-iteration
            for \(k=1\) to \(\left|R_{\pi}^{r s}\right|+1\) do
                \(D_{k} \leftarrow \operatorname{GetDTRSs}\left(k, U,\left|R_{\pi}^{r s}\right|\right) ; g=0 ;\)
                while \(g<\left|D_{k}\right|\) do
                \(g^{++}, H \leftarrow \mathrm{HTs}\) of tokens in \(d_{g}\);
                if \(H\) violates recursive \(\left(c_{k}, \ell_{k}\right)\)-diversity then
                    go to Line 4 to the next for-iteration
```

            return rs;
    optimal one. Since $t_{\tau}$ has to be contained in the new RS, we let the set of candidate tokens $\sigma$ for the new RS be $\sigma=T \backslash t_{\tau}$ (line 1 ). We search an eligible RS following the ascending order of the sizes of RSs, and $i$ indicates the number of mixins in the RS (lines 2-23). In each iteration of lines 2-23, there is a set $C R^{i}$ of different RSs, where each $c r \in C R^{i}$ contains $i$ different tokens of $\sigma$ (line 3). For each $c r$, the corresponding candidate RS is $r s=t_{\tau} \cup c r$, and the related RS set is $R_{\pi}^{r s}$ (line 5). Let $H$ be the set of HTs of tokens in $r s$ (line 6). $H$ should satisfy the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity, otherwise, we go to check the next RS (lines 7-8). If $H$ is valid, we go to check if $r s$ satisfies the non-eliminated constraint (lines 9-16). We retrieve all valid token-RS combinations over $R_{\pi}^{r s} \cup r s$ as $U$ (line 9). For each $r_{k} \in R_{\pi}^{r s} \cup r s$, we check if there are some tokens in $r_{k}$ can be eliminated by traversing $U$ (lines 10-16). ST stores all consumed token of $r_{k}$ in any token-RS combination $u_{j} \in U$, and $w_{j, k}$ indicates the consumed token of $r_{k}$ in $u_{j}$. If it satisfies the non-eliminated constraint, we check if, for any $r_{k} \in R_{\pi}^{r s} \cup r s$, its DTRSs all satisfy the recursive $\left(c_{k}, \ell_{k}\right)$-diversity, where $r_{\left|R_{\pi}^{T}\right|+1}=r s$ (lines 17-22). We retrieve all DTRSs of $r_{k}$ by the GetDTRSs procedure (line 18). Let $H$ be the set of HTs of tokens in $d_{g}$ (line 20). If $H$ violates recursive ( $c_{\tau}$, $l_{\tau}$ )-diversity, we go to check the next RS (lines 21-22). If $r s$ satisfies all constraints, we return it as the result (line 23).
The GetDTRSs Procedure. Algorithm 3 illustrates the GetDTRSs procedure. We set the candidate DTRS sets $D^{*}$ and $D^{\#}$ as empty sets. Let $D H$ be the set of determined HT of consumed tokens of $r_{k}$ by DTRSs in $D^{*}$. Given a DTRS $d_{i}$, we can determine that the HT of

```
Algorithm 3: GetDTRSs
    Input: a integer \(k\), a set of token-RS combinations \(U\), and a integer \(n\)
    Output: a set of DTRSs whose size is \(k\)
    \(D^{*}=D^{\#}=\emptyset, D H=\emptyset ;\)
    foreach \(u \in U\) do
        \(p^{*} \leftarrow\) the token-RS pair of \(r_{k}\) in \(u, u^{*} \leftarrow u \backslash p^{*} ;\)
        for \(i=1\) to \(n d o\) do
            \(D^{*}=D^{*}+\) the set of combinations of choosing \(i\) pairs from \(u^{*}\);
            for \(j=1\) to \(C(n, i) d o\) do
                \(D H=D H+\) the historical transaction of the token in \(p^{*}\);
    foreach \(d_{i} \in D^{*}\) do
        for \(j=1\) to \(\mid U /\) do
            \(p^{\#} \leftarrow\) the token-RS pair of \(r_{k}\) in \(u_{j}\)
            if \(d_{i} \notin u_{j}\) then
                continue;
                if the \(H T\) of the token in \(p^{\#}\) is not equal to \(d h_{i}\) then
                go to Line 8 to next for-iteration
        \(D^{\#}=D^{\#}+d_{i} ;\)
    remove the super set in \(D^{\#}\);
    return \(D^{\#}\);
```

the consumed token in $r_{k}$ is $d h_{i}$. For each token-RS combination $u$ in $U$ and a size $i$, we enumerate all combinations of $i$ pairs in $u$ (lines 2-7). Let $p^{*}$ be the token-RS pair of $r_{k}$ in $u$ and $u^{*}$ be the $u$ excluding $p^{*}$ (line 3). We add these candidate DTRSs into $D^{*}$ (line 5) and their corresponding $d h s$ in $D H$ are the HTs of the tokens in $p^{*}$ (lines 6-7). These candidate DTRSs in $D^{*}$ may not be the "true" DTRSs as Definition 2 defined. For each $d_{i}$ in $D^{*}$, we check all token-RS combinations in $U$ (lines 9-14). If a DTRS $d_{i}$ is in a $u_{j}$ but the HT of the consumed token of $r_{k}$ in $u_{j}$ is not $d h_{i}, d_{i}$ is a "fake" DTRS (lines 13-14), otherwise, we add it to $D^{\#}$ (line 15). Since the supersets of "true" DTRSs are also added to $D^{\#}$, we remove them and return the updated $D^{\#}$ as the result (line 16-17).
Time Complexity. Let $m=\left|R_{\pi}^{r s}\right|=O(n)$. There are $\sum_{i=\ell_{\tau}-1}^{n-1} C(n-$ $1, i)=O\left(2^{n}\right)$ possible RSs in $\bigcup_{i=\ell_{\tau}-1}^{|\sigma|} C R_{i}$, where $C(n-1, i)$ is the number of combinations of selecting $i$ elements from $n-1$ elements [23]. For each $r s$, the cardinality of $U$ is $O\left(\prod_{r_{i} \in R_{\pi}^{r s} \cup r s}\left|r_{i}\right|\right)$ $=O\left(n^{m}\right)$. In the GetDTRSs procedure, the time complexity of getting $D^{*}$ is $O\left(n^{m}\right) \cdot \sum_{i=1}^{m} C(m, i)=O\left(n^{m} 2^{m}\right)$ and the cardinality of $D^{*}$ is $O\left(n^{m} 2^{m}\right)$. Thus, the time complexity of getting $D^{\#}$ is $O\left(n^{2 m} 2^{m}\right)$. Thus, the total time complexity of the BFS approach is $O\left(m \cdot 2^{m+n} \cdot n^{2 m}\right)=O\left(n^{n}\right)$.

## 6 PRACTICAL SOLUTIONS FOR DA-MS

Although the BFS approach can get the optimal solution, its time complexity is unacceptable for most real-world applications. Besides, users sometimes cannot find a new RS satisfying the immutability constraint. For example, $T=\left\{t_{1}, \cdots, t_{4}\right\}$, where four tokens are from four different HTs. The first user uses $r_{1}=\left\{t_{1}, t_{2}, t_{3}\right\}$ with recursive (1,2)-diversity to consume $t_{1}$. The second user uses $r_{2}=\left\{t_{1}, t_{2}, t_{4}\right\}$ with recursive (2,3)-diversity to consume $t_{4}$. The third user uses $r_{3}=\left\{t_{1}, t_{2}, t_{3}, t_{4}\right\}$ with recursive (1,3)-diversity to consume $t_{3}$. Then, the fourth user cannot find an eligible RS to consume $t_{2}$ satisfying the immutability constraint.

To efficiently solve the DA-MS problem, we have three challenges: (1) how to efficiently verify if DTRSs of a RS satisfy the anonymity requirement; (2) how to guarantee that a user can always find a RS satisfying the immutability constraint; and (3) how to pick a desired set of mixins with the minimal cardinality under the constraints in Definition 5. In this section, we propose two practical configurations to tackle the first two challenges. To solve the third challenge, we design two approximation algorithms, the Progressive Algorithm and the Game-theoretic Algorithm.

### 6.1 Practical Configurations

In this section, we propose two practical configurations to help solve the first two challenges. These two configurations guide the mixin selection in the Step 1 of a RS scheme. In the Step 3 of a RS scheme, verifiers would check if a RS satisfies these two configurations. The first practical configuration is that each new RS should be the superset of some RS in $R_{\pi}^{T}$ and be disjoint with the other RSs in $R_{\pi}^{T}$, where $R_{\pi}^{T}$ is the set of RSs containing tokens in $T$. With this configuration, we have some special RSs in $R_{\pi}^{T}$, called as super $R S$.
Definition 7 (A Super Ring Signature). Given a related RS set $R_{\pi}^{T}$, a RS $r_{i}$ is a super RS if for any $r_{j}$ in $R_{\pi}^{T}$ that is proposed after $r_{i}, r_{j}$ is not the super set of $r_{i}$. Let $v_{i}$ be the number of RSs in $R_{\pi}^{T}$ that are the subsets of $r_{i}$.

Let $S_{\pi}^{T}=\left\{s_{1}, \cdots, s_{n}\right\}$ be the set of super RSs in $R_{\pi}^{T}$. Some tokens in $T$ may not be contained in any RS in $R_{\pi}^{T}$. We call these tokens as fresh tokens. Let $F_{\pi}^{T}=\left\{f_{1}, \cdots, f_{n}\right\}$ be the set of fresh tokens in $T$.
Definition 8 (Fresh Token). Given a related RS set $R_{\pi}^{T}$, if a token $t$ is not contained in any RS in $R_{\pi}^{T}, t$ is a fresh token.

For example, $r_{1}=\left\{t_{1}, t_{2}\right\}$ is proposed at time $\pi, r_{2}=\left\{t_{1}, t_{2}, t_{3}\right\}$ is proposed at time $\pi+1$, and $r_{3}=\left\{t_{4}, t_{5}\right\}$ is proposed at time $\pi+2$. Suppose $T=\left\{t_{1}, \cdots, t_{6}\right\}$ and $R_{\pi}^{T}=\left\{r_{1}, r_{2}, r_{3}\right\}$. Then, $r_{2}$ and $r_{3}$ are two super RSs while $r_{1}$ is not a super RS. The subset number $v_{i}$ of $r_{2}$ is 2 (i.e., $r_{1}$ and $r_{2}$ ). Besides, $t_{6}$ is a fresh token.

Thus, by the first practical configuration, each new RS is made up of some super RSs and some fresh tokens. Then, we show how to get the token set of a DTRS of a RS.

Theorem 6.1. In a blockchain system, a new RS is made up of some super RSs and some fresh tokens. For a related RS set R, let the super RS of a RS $r_{i} \in R$ be $r_{i *}$ and its subset number be $v_{i^{*}}$. Let $\widetilde{T_{i, j}}$ be the set of tokens in $r_{i}$ whose historical transaction is $h_{j}$. Let $\psi_{i, j}=r_{i} \backslash \widetilde{T_{i, j}}$. If $v_{i^{*}}<\left|r_{i}\right|-\left|\widetilde{T_{i, j}}\right|+1$, there is no DTRS of $r_{i}$ that can determine the historical transaction of the consumed token of $r_{i}$ is $h_{j}$. Otherwise, $\psi_{i, j}$ is the token set of a DTRS of $r_{i}$ that can determine the $H T$ of the consumed token of $r_{i}$ is $h_{j}$.

Proof. To determine the historical transaction of the consumed token of $r_{i}$ is $h_{j}, r_{i} \backslash \widetilde{T_{i, j}}$ should be eliminated. If $v_{i^{*}}<\left|r_{i}\right|-\left|\widetilde{T_{i, j}}\right|+1$, by the Pigeonhole principle [22], there is at least one token in $\widetilde{T_{i, j}}$ have not been consumed. Thus, if $v_{i^{*}}<\left|r_{i}\right|-\left|\widetilde{T_{i, j}}\right|+1$, there is no DTRS of $r_{i}$ that can determine the historical transaction of the consumed token of $r_{i}$ is $h_{j}$. Let $d$ be a DTRS that determine the historical transaction of the consumed token of $r_{i}$ is $h_{j}$ and $t^{*}$ is the set of tokens in $d$. Thus, $t^{*} \subseteq r_{i} \backslash \widetilde{T_{i, j}}$. Let $u^{*}$ be a set of token-RS pairs whose token's union is $r_{i} \backslash \widetilde{T_{i, j}}$. Assume $\bar{p}=(\bar{t}, \bar{r})$ is a token-RS
pair in $u^{*}$ and not in $d$. Since $\bar{t} \in r_{i}, \bar{t}$ can be the consumed token of $r_{i}$. Since $\bar{t} \notin \widetilde{T_{i, j}}$, when $\bar{t}$ is the consumed token of $r_{i}$, the historical transaction of $r_{i}$ is not $h_{j}$, which violates the definition of a DTRS. Thus, the assumption does not hold. Thus, if $v_{i^{*}} \geq\left|r_{i}\right|-\left|\widetilde{T_{i, j}}\right|+1$, $\psi_{i, j}$ is the token set of a DTRS of $r_{i}$ that can determine the historical transaction of the consumed token of $r_{i}$ is $h_{j}$.

Thus, to verify if all DTRSs of a RS satisfy the recursive diversity, we just need to verify if each $\psi_{i, j}$ satisfies the recursive diversity, which can be solved in polynomial time. Thus, the first challenge is solved. Besides, we prove that if the cardinality of the side information of an adversary is smaller than a threshold, she/he cannot confirm the historical transaction of the consumed token of a RS.

Theorem 6.2. If the cardinality of the side information of an adversary is smaller than $\left|r_{i}\right|-q_{M}$, she/he cannot confirm the HT of the consumed token of a RS $r_{i}$, where $q_{M}$ is the number of times that the most-frequent HT of tokens appears in $r_{i}$.

Proof. By Theorem 6.1, to confirm the historical transaction of the spent token of $r_{i}$ is $h_{j}$, an adversary needs to know at least $r_{i}$ $-\left|\widetilde{T_{i, j}}\right|$ token-RS pairs. Since $q_{M}$ is the number of times the mostfrequent HT of tokens in $r_{i}$, if the cardinality of the side information of an adversary is smaller than $\left|r_{i}\right|-q_{M}$, the adversary cannot confirm the historical transaction of the consumed token of $r_{i} . \quad \square$

By Theorem 6.2, to increase the threshold of the cardinality of an adversary's side information, when the size of a RS is fixed, a user should decrease the number of times the most-frequent HT of tokens in the RS. It exactly meets the idea of the recursive ( $c, \ell$ )-diversity (e.g, let the set of HTs are not dominated by the most-frequent HT), which provides the motivation for us to use the recursive $(c, \ell)$-diversity principle to measure the anonymity of a RS. Besides, we prove that, if before observing a new RS $r$, an adversary cannot confirm if $t$ is the spent token of a RS $r^{\prime}$, after observing $r$, she/he still cannot confirm if $t$ is the spent token of $r^{\prime}$.

Theorem 6.3. Let $t$ be a token in a $R S r^{\prime}$. Suppose before observing a new RS $r$, an adversary cannot confirm ift is the spent token of $r^{\prime}$. The adversary still cannot confirm ift is the spent token of $r^{\prime}$ after observing $r$.

Proof. Suppose $D_{b}^{\prime}$ is the set of DTRSs of $r^{\prime}$ before observing a new RS $r$ and $D_{a}^{\prime}$ is the set of DTRSs of $r^{\prime}$ after observing $r$. By the first practical configuration, $r \cap r^{\prime}=\emptyset$ or $r^{\prime} \subseteq r$. By Theorem 6.1, if $r \cap r^{\prime}=\emptyset, D_{b}^{\prime}=D_{a}^{\prime}$. Thus, if $r \cap r^{\prime}=\emptyset$, after observing $r$, the adversary still cannot confirm if $t$ is the spent token of $r^{\prime}$. If $r^{\prime} \subseteq r$, $t \in r$. Since the adversary cannot confirm if $t$ is the spent token of $r^{\prime}$, the adversary also cannot confirm if $t$ is the spent token of $r$. Thus, $t$ at least may be consumed in $r^{\prime}$ or $r$. Thus, the adversary still cannot confirm if $t$ is the spent token of $r^{\prime}$ after observing $r$.

For the second challenge, we propose the second practical configuration. Specifically, if a user wants to generate a RS and guarantee that each DTRS satisfies the recursive $(c, \ell)$, she/he should ask the HT set of tokens of new RS satisfies the recursive $(c, \ell+1)$-diversity.

Theorem 6.4. If the HT set of tokens ofr satisfies the recursive $(c, \ell+1)$-diversity, the HT set of tokens of any one of its DTRSs must satisfy the recursive $(c, \ell)$-diversity.

```
Algorithm 4: Progressive Algorithm.
    Input: the consumed token \(t_{\tau}\), a mixin universe \(T\), the realted RS set
            \(R_{\pi}^{T}\), and the required diversity constraint \((c, \ell)\)
    Output: An eligible RS \(r_{\tau}\)
    get the super RS set \(S\) and the fresh token set \(F\), get \(X, r_{\tau}=x_{\tau}\), get \(H\);
    while \(|H|<\ell_{\tau}\) do
        calculate \(\alpha_{i}\) for each \(x_{i} \in X\);
        \(r_{\tau} \leftarrow r_{\tau}+x_{i}\) in \(X\) wit the minimal \(\alpha_{i}\), update \(H\) and \(X\);
    while \(|H|\) violates the recursive \(\left(c_{\tau}, \ell_{\tau}\right)\)-diversity requirement do
        calculate \(\beta_{i}\) for each \(x_{i} \in X\);
        \(r_{\tau} \leftarrow r_{\tau}+x_{i}\) in \(X\) wit maximal \(\beta_{i}\), update \(H\) and \(X\);
    return \(r_{\tau}\)
```

Proof. Suppose the tokens of $r$ come from $\theta$ historical transactions. Let $q_{r, i}$ be the number of times the $j^{t h}$ most-frequent HT of tokens in $r$. Suppose $d$ is a DTRS of $r$. By Theorem 6.1, the tokens of $d$ come from $\theta-1$ historical transactions. Let $q_{d, i}$ be the number of times the $j^{\text {th }}$ most-frequent HT of tokens in $d$. Since the HT set of tokens of $r$ satisfies the recursive $(c, \ell+1)$-diversity, $q_{r, 1}<c \cdot\left(q_{r, \ell+1}+\cdots q_{r, \theta}\right)$. Since $q_{d, 1} \leq q_{r, 1}$ and $q_{r, \ell+1}+\cdots q_{r, \theta}$ $\leq q_{d, \ell}+\cdots q_{d, \theta-1}, q_{d, 1}<c \cdot\left(q_{d, \ell}+\cdots q_{d, \theta-1}\right)$. Thus, if the HT set of tokens of $r$ satisfies the recursive $(c, \ell+1)$-diversity, the HT set of tokens of its any one of its DTRSs satisfies the recursive $(c, \ell)$-diversity.

With Theorem 6.4, if the HT set of tokens of a RS $r$ satisfies the recursive $\left(c_{\tau}, \ell_{\tau}+1\right)$-diversity, other users always can find new RSs to maintain $r$ 's recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity. Thus, the second challenge is solved.

### 6.2 Progressive Algorithm

In this subsection, to tackle the third challenge and approximately get a solution of the DA-MS problem, we propose the Progressive Algorithm. It first greedily finds a set of mixins coming from more than $\ell_{\tau}$ HTs. Then it greedily changes the candidate RS to satisfy the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity.

Algorithm 4 illustrates the pseudo-code of our Progressive Algorithm. We first retrieve the set of super RSs and the set of fresh tokens (line 1). With the first configuration in Subsection 6.1, each RS is consisted of some super RSs and some fresh tokens. Thus, we consider a super RS or a fresh token as a module which can be selected in $r_{\tau}$ and $X$ is the set of modules (line 1). The module $x_{\tau}$ is the super RS which contains $t_{\tau}$ or the $t_{\tau}$ itself when $t_{\tau}$ is a fresh token (line 1). Let $H$ be the set of HTs outputting tokens in $r_{\tau}$ and $H_{i}$ is the set of HTs outputting tokens in $x_{i}$. We first let the tokens in $r_{\tau}$ come from at least $\ell_{\tau}$ different HTs (line 2-4). For each $x_{i} \in X$, we calculate $\alpha_{i}=\frac{\left|x_{i}\right|}{\min \left\{\ell_{\tau}-|H|,\left|H_{i} \backslash\left(H \cap H_{i}\right)\right|\right\}}$ (line 3). We add the module $x_{i}$ with the minimal $\alpha_{i}$ to $r_{\tau}$ and update $H$ and $X$ (line 4). Next, we let $H$ satisfy the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity (line 5-7). Let $t r_{i}=r_{\tau} \cup x_{i}$ and $t H_{i}$ be the set of HTs outputting tokens in $t r_{i}$. Let $q_{i, j}$ be the number of times the $j^{t h}$ most-frequent HT of tokens in $t r_{i}$. For each $x_{i}$ in $X$, we calculate $\beta_{i}=\frac{\delta-\delta_{i}}{\left|x_{i}\right|}$, where $\delta=q_{\tau, 1}-c_{\tau} \cdot\left(q_{\tau, \ell_{\tau}}+\cdots+q_{\tau,|H|}\right), \delta_{i}=q_{i, 1}-c_{\tau} \cdot\left(q_{i, \ell_{\tau}}+\cdots+q_{i,\left|t H_{i}\right|}\right)$, and $\left|x_{i}\right|$ is the size of $x_{i}$ (line 6). We greedily add the module whose $\beta_{k}$ is maximal and update $H$ and $X$ (line 7). Finally, we return $r_{\tau}$ (line 8).

```
Algorithm 5: Game-theoretic Algorithm.
    Input: the consumed token \(t_{\tau}\), a mixin universe \(T\), the realted RS set
            \(R_{\pi}^{T}\), and the required diversity constraint \((c, \ell)\)
    Output: An eligible RS \(r_{\tau}\)
    get the super RS set \(S\) and the fresh token set \(F\), get \(A, r_{\tau}=a_{\tau}\), get \(H\);
    while \(|H|<\ell_{\tau}\) do
        calculate \(\gamma_{i}\) for each \(a_{i} \in A\);
        set the strategy of \(a_{i}\) with the minimal \(\gamma_{i}\) as \(\phi\), update \(r_{\tau}\) and \(H\);
    5 repeat
        foreach play \(a_{i} \in A\) do
            \(e_{i}=\phi ;\)
            if the cost of \(\bar{\phi}\) is lower then
                \(e_{i}=\bar{\phi} ;\)
            update \(r_{\tau}\) and \(H\);
    until reaching a Nash equilibrium
    return \(r_{\tau}\)
```

Example 3. There are four super $R S s, s_{1}=\left\{t_{1}, \cdots, t_{6}\right\}, s_{2}=$ $\left\{t_{7}, \cdots, t_{10}\right\}, s_{3}=\left\{t_{11}, t_{12}\right\}$, and $s_{4}=\left\{t_{13}, \cdots, t_{15}\right\}$. The tokens $t_{1}$, $t_{2}, t_{7}$ and $t_{8}$ are from $h_{1}$. The tokens $t_{3}, t_{4}$, and $t_{9}$ are from $h_{2}$. The tokens $t_{5}, t_{13}$ and $t_{14}$ are from $h_{3}$. The tokens $t_{6}$ and $t_{10}$ are from $h_{6}$. The token $t_{11}$ and $t_{15}$ are from $h_{4}$. The token $t_{12}$ are from $h_{5} . A$ user wants to generate a RS to consume $t_{11}$ with the recursive (1,4)diversity. Then, $x_{\tau}=s_{3}$. In the first iteration of the first while-loop, we add $s_{2}$ to $r_{\tau}$ since $\alpha_{2}$ is minimal. After the first while-loop, we get $r_{\tau}=s_{3} \cup s_{2}$. In the first iteration of the second while-loop, we add $s_{4}$ to $r_{\tau}$, since $\beta_{4}=\frac{1}{3}$ and $\beta_{1}=-\frac{1}{6}$.

Theorem 6.5. The Progressive Algorithm's approximation ratio is $\epsilon+\frac{q_{M} \cdot z_{M}}{10^{-\gamma}}$, where $\epsilon=\sum_{i=1}^{\ell_{\tau}} \frac{1}{i}, q_{M}$ is the number of times the mostfrequent HT of tokens in $T, z_{M}$ is the maximal size of a super RS in $R_{\pi}^{T}$, and $\gamma$ is the minimum integer such that $10^{\gamma} \cdot \epsilon_{\tau}$ is an integer.

Proof. Suppose OPT is the size of the optimal result of the input DA-MS problem instance. Let $z^{*}$ be the minimal size of a selection from $X$ whose tokens are from at least $\ell_{\tau}$ different HTs. Let $q_{\text {min }}$ be the number of times the most-infrequent HT of tokens in $T$. Thus, $O P T \geq z^{*} \geq \ell_{\tau} q_{\text {min }}$. Let $z^{\#}$ be the size of $r_{\tau}$ when the first whileloop terminates. By [24], $\frac{z^{*}}{z^{*}} \leq \epsilon$. Let $x_{k}$ be the last module which is added to $r_{\tau}$ and $z_{r}$ is the size of $r_{\tau}$ that the Progressive Algorithm returns. In each iteration of the second while-loop, $\delta$ decrease at least $10^{-\gamma}$. Since $\left|x_{k}\right| \leq z_{M}$ and $\delta \leq q_{M}, z_{r} \leq z^{\#}+\frac{q_{M} \cdot z_{M}}{10^{-\gamma}}$. Thus, $\frac{z_{r}}{O P T} \leq \frac{z^{\#}}{O P T}+\frac{q_{M} \cdot z_{M}}{10^{-\gamma} \cdot \ell_{\tau} \cdot q_{\min }} \leq \epsilon+\frac{q_{M} \cdot z_{M}}{10^{-\gamma}}$.

In Example 3, $q_{M}=4, q_{\text {min }}=1, z_{M}=6$, and $\gamma=0$. These parameters are determined by the problem instance. When $\ell_{\tau}$ is larger, it is harder to satisfy the diversity constraint. Then, the first while-loop (lines 2-4) would get a RS with higher size, which is denoted by the first term of the approximation ratio (i.e., $\epsilon$ ). When $c_{\tau}$ is smaller, it is harder to satisfy the diversity constraint. Then, the second while-loop (lines 5-7) would get a RS with higher size, which is represented by the second term of the approximation ratio (i.e., $\left.\frac{q_{M} \cdot z_{M}}{10^{-\gamma}}\right)$.
Time Complexity. Let the cardinality of the mixin universe be $n$, i.e., $n=|T|$. In each iteration of the first while-loop, $|H|$ increases by at least 1 . Thus, the round of the first while-loop is $O\left(\ell_{\tau}\right)$. For each round, there are $O(n)$ modules in $X$. For each module, the


Figure 3: The Distribution of the Number of Tokens.
calculation of $\alpha_{k}$ is $O(n)$. Compared with $n, \ell_{\tau}$ usually can be ignored. Thus, the time complexity of the first while-loop is $O\left(n^{2}\right)$. In each iteration of the second while-loop, $\delta$ decreases by at least $10^{-\gamma}$. Thus, the round of the second while-loop is $O\left(\frac{q_{M}}{10^{-\gamma}}\right)$. For each module, the calculation of $\beta_{k}$ is $O(n)$. Compared with $n, \frac{q_{M}}{10^{-\gamma}}$ usually can be ignored. Thus, the time complexity of the second while-loop is $O\left(n^{2}\right)$. Thus, the total time complexity is $O\left(n^{2}\right)$.

### 6.3 Game-theoretic Algorithm

Since the Progressive Algorithm greedily picks modules (super RSs or fresh tokens), it may result in local optimal solutions. In this subsection, we develop a game-theoretic algorithm, where super RSs and fresh tokens will be considered as players to find an eligible $r_{\tau}$ whose size is as small as possible until the game reaches a Nash equilibrium [25]. In this section, we prove that the game can reach a Nash equilibrium within polynomial time and the size of $r_{\tau}$ is theoretically guaranteed when a Nash equilibrium is converged.

Algorithm 5 illustrates the pseudo-code of our Game-theoretic Algorithm. We first retrieve the set of super RSs and the set of fresh tokens (line 1). With the first configuration in Subsection 6.1, each new RS should be made up of some super RSs and some fresh tokens. Thus, we consider a super RS or a fresh token as a player who competes the right to be picked in $r_{\tau}$ with other players. Each player has two strategies, $\phi$ and $\bar{\phi}$, which indicate being selected and being not selected in the new RS, respectively. The player $a_{\tau}$ is the super RS which contains $t_{\tau}$ or the $t_{\tau}$ itself when $t_{\tau}$ is a fresh token (line 1). Since $a_{\tau}$ has to been contained in $r_{\tau}$, we remove it from $A$. Suppose $H$ is the set of HTs outputting tokens in $r_{\tau}$ and $H_{i}$ is the set of HTs outputting tokens in $a_{i}$. Then, we first let the tokens in $r_{\tau}$ be from at least $\ell_{\tau}$ different HTs (line 2-4). For each player $a_{i} \in A$, we calculate $\gamma_{i}=\frac{\left|a_{i}\right|}{\min \left\{\ell_{\tau}-|H|,\left|H_{i} \backslash\left(H \cap H_{i}\right)\right|\right\}}$ (line 3). We add the player $a_{i}$ with the minimal $\gamma_{i}$ to $r_{\tau}$ and update $H$ (line 4). Next, we let $H$ satisfy the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity (line 5-11). For each player, we calculate the cost of two strategies when other users' strategies are given and chose the strategy with the minimal cost (line 6-10). When the cost of two strategies are the same, we chose the strategy $\phi$ (line 7). Specifically, for each player $a_{i}$, the cost function of strategies is $c_{i}\left(e_{i}, \overline{e_{i}}\right)=$ $\left\{\begin{array}{ll}\left|\widetilde{\tau}_{\tau}\right| & H \text { satisfies the recursive }\left(c_{\tau}, \ell_{\tau}\right) \text {-diversity } \\ \infty, & \text { otherwise }\end{array}\right.$, where $\overline{e_{i}}$ is the set of strategies of players in $A$ except $a_{i}$, and $\widetilde{r_{\tau}}$ is the RS generated by $e_{i}$ and $\overline{e_{i}}$ (line 8). Finally, we return $r_{\tau}$ as the result of the Game-theoretic Algorithm (line 12).

In Example 3, by the Game-theoretic Algorithm, the RS is $r_{\tau}=$ $s_{1} \cup s_{3}$. Specifically, after the first while-loop, we get $r_{\tau}=s_{3} \cup s_{2}$.


Figure 4: The running time of the $i^{\text {th }} \mathbf{R S}$ by the $T M_{-} B$ Approach. Then, for the player $s_{1}$, the $c_{1}\left(\phi, \overline{e_{1}}\right)=c_{1}\left(\bar{\phi}, \overline{e_{1}}\right)=\infty$. Thus, $e_{1}=\phi$ and $r_{\tau}=s_{1} \cup s_{2} \cup s_{3}$. For the player $s_{2}$, the $c_{2}\left(\bar{\phi}, \overline{e_{2}}\right)=\frac{8}{3}$ and $c_{2}\left(\phi, \overline{e_{2}}\right)=\infty$. Thus, $e_{2}=\bar{\phi}$ and $r_{\tau}=s_{1} \cup s_{3}$.

Next, we prove that the Game-theoretic Algorithm can converge on a Nash equilibrium within polynomial time.

Theorem 6.6. Game-theoretic Algorithm can converge on a Nash equilibrium within $O\left(n^{3}\right)$, where $n=|T|$.

Proof. Define the potential function of the game as $\Phi(\tilde{e})=$ $\left\{\begin{array}{l}\left|\widetilde{r}_{\tau}\right| \\ |A|\end{array} \quad H\right.$ satisfies the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity, $\infty$ otherwise , where $\tilde{e}$ is the set of strategies of players in $A, \widetilde{r_{\tau}}$ is the RS which is generated according to $\tilde{e}$, and $H$ is the set of HTs outputting tokens in $r_{\tau}$. Thus, $\forall a_{i} \in A, c_{i}\left(e_{i}, \overline{e_{i}}\right)-c_{i}\left(e_{i}^{\prime}, \overline{e_{i}}\right)=\Phi(\tilde{e})-\Phi\left(\tilde{e}^{\prime}\right)$, where $\tilde{e}=e_{i}+\overline{e_{i}}$ and $\tilde{e}^{\prime}=e_{i}^{\prime}+\overline{e_{i}}$. By [26], when $\forall a_{i} \in A, c_{i}\left(e_{i}, \overline{e_{i}}\right)-c_{i}\left(e_{i}^{\prime}, \overline{e_{i}}\right)=$ $\Phi(\tilde{e})-\Phi\left(\tilde{e}^{\prime}\right)$ and the strategy set of each player is finite, the game can converge on a Nash equilibrium. Since in each iteration, $\Phi(\tilde{e})$ decreases by at least $\frac{1}{A}$ and $\Phi(\tilde{e}) \leq \frac{n}{|A|}$, the number of iteration is $O(n)$. For each iteration, we calculate the cost function of $O(n)$ players and the time complexity of each calculation is $O(n)$. Thus, the total time complexity is $O\left(n^{3}\right)$.

Then, we prove the approximation ratio of the Game-theoretic Algorithm. Suppose OPT is the cost of the optimal solution to the problem where the the sum of each user's cost function is minimized. As standard measurements, price of stability (PoS) and price of anarchy $(P o A)$ are often used to evaluate the quality of an equilibrium [27-29]. Specifically, PoS of a game is the ratio of the minimized cost when it reaches an equilibrium to the OPT. Besides, PoA of a game is the ratio of the maximized cost when it reaches an equilibrium to the $O P T$.

Theorem 6.7. The PoS is bounded by 1 and the PoA is bounded by $q_{M} \cdot\left(1+\frac{1}{c_{\tau} \cdot \ell_{\tau}}\right)+\frac{z_{M}}{\ell_{\tau}}$, where $q_{M}$ is the number of times that the most-frequent $H T$ of tokens appears in $T$, and $z_{M}$ is the maximal size of a super RS in $R_{\pi}^{T}$.

Proof. By the definition of the potential function in the proof of Theorem 6.6, when a Nash equilibrium is converged, $\Phi(\tilde{e})=\frac{\left|\widetilde{r}_{\tau}\right|}{|A|}$. Let OPT be the size of the optimal result, denoted by $r_{o}$, of the input DA-MS problem instance and let $\tilde{e}^{*}$ be the corresponding strategy set of players in $A$. Let $\tilde{e}^{\#}$ be the strategy set of players in $A$ that yields the minimum of $\Phi(\tilde{e})$. In other words, when the strategy set of players in $A$ is $\tilde{e}^{\#}$, the game reaches the best Nash equilibrium. Let $r^{\#}$ be the RS when the strategy set of players is $\tilde{e}^{\#}$. Thus, $O P T=|A| \cdot \Phi\left(\tilde{e}^{*}\right) \geq|A| \cdot \Phi\left(\tilde{e}^{\#}\right)=\left|r^{\#}\right|$. Thus, $P o S=\frac{\left|r^{\#}\right|}{O P T} \leq 1$. Let $r_{c}$ be a RS that the Game-theoretic Approach converges, $a_{k}$ be the last player who changes her/his strategy before the converging,

Table 2: Experimental Settings (Real).

| Parameters | Values |
| :---: | :--- |
| the $c_{\tau}$ of the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity | $0.2,0.4, \mathbf{0 . 6}, 0.8,1$ |
| the $\ell_{\tau}$ of the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity | $20,30, \mathbf{4 0}, 50,60$ |

Table 3: Experimental Settings (Synthetic).

| Parameters | Values |
| :--- | :--- |
| the size of each super RS $\left\|s_{i}\right\|$ | $[1,10],[5,15],[\mathbf{1 0 , 2 0 ]},[15,25],[20,30]$ |
| the number of super RSs $\|S\|$ | $10,30, \mathbf{5 0}, 70,90$ |
| the number of fresh tokens $\|F\|$ | $0,5, \mathbf{1 0}, 15,20$ |
| the variance $\sigma$ of tokens' distribution | $8,10, \mathbf{1 2}, 14,16$ |

and $r_{k}$ be the RS before the last change of $r_{c}$. Thus, $\left|r_{c}\right| \leq\left|r_{k}\right|+\left|a_{k}\right|$. Denote $q_{c, i}$ as the number of times the $i^{t h}$ most-frequent HT of tokens in $r_{c}$ and $q_{k, i}$ as the number of times the $i^{t h}$ most-frequent HT of tokens in $r_{k}$. Since $r_{k}$ violates the recursive ( $c_{\tau}, \ell_{\tau}$ )-diversity, $q_{k, \ell_{\tau}}+\cdots+q_{k,\left|H_{k}\right|} \leq \frac{q_{k, 1}}{c_{\tau}} \leq \frac{q_{M}}{c_{\tau}}$, where $H_{k}$ is the set of different HTs outputting tokens in $r_{k}$. Thus, $\left|r_{c}\right| \leq\left|r_{k}\right|+\left|a_{k}\right|=q_{k, 1}+$ $\cdots q_{k,\left|H_{k}\right|}+\left|a_{k}\right| \leq q_{M} \cdot\left(\ell_{\tau}-1\right)+\frac{q_{M}}{c_{\tau}}+z_{M}$. Let $q_{\text {min }}$ be the number of times the most-infrequent HT of tokens in $T$. Since $r_{o}$ satisfies the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity constraint, $O P T \geq \ell_{\tau} \cdot q_{\text {min }}$. Thus, PoA $=\frac{\left|r_{c}\right|}{O P T} \leq \frac{q_{M} \cdot\left(\ell_{\tau}-1\right)+\frac{q_{M}}{c_{\tau}}+z_{M}}{\ell_{\tau} \cdot q_{\text {min }}} \leq q_{M} \cdot\left(1+\frac{1}{c_{\tau} \cdot \ell_{\tau}}\right)+\frac{z_{M}}{\ell_{\tau}}$.

When $q_{M}$ is higher, it is harder to satisfy the diversity constraint and the algorithm would pick more tokens, which would increase the approximation ratio. Because of the first practical configuration in Subsection 6.1, each super RS cannot be partly picked. Thus, when $z_{M}$ is larger, the final RS which is generated by our algorithms would be larger, which would increase the approximation ratio.

## 7 EXPERIMENTAL STUDY

### 7.1 Experiment Configuration

Data Sets. We test our proposed practical approaches over real data sets as well as synthetic data sets. For real data sets, we retrieve the tokens in the blocks between 2,028,242 and 2,028,273 from the Monero System, which were generated in one hour. There are 285 transactions and 633 tokens in these blocks. Since in Monero System, most RS's size is 11 , we generate 57 super RSs and 6 fresh tokens. For each super RSs, it randomly selects 11 tokens. Figure 3 shows the distribution of the number of tokens in a transaction. Most transactions output two tokens. In other words, the distribution of HTs of tokens is almost uniform, and in a RS most $q_{i}$ does not exceed 2 . If we set $c_{\tau}$ large and set $\ell_{\tau}$ small, the recursive $\left(c_{\tau}, \ell_{\tau}\right)$ diversity constraint would be very relax. In experiments over the real data sets, we vary $c_{\tau}$ from 0.2 to 1 and vary $\ell_{\tau}$ from 20 to 60 .

To further examine the effects of the distribution of HTs of tokens, the number of super RSs, the number of fresh tokens, and the size of each super RS, we generate the synthetic data sets and run the experiments. For synthetic data sets, we generate $|S|$ super RSs where $|S|$ is varied from 10 to 90 . We uniformly set the size of a super RS within the range $\left[s^{-}, s^{+}\right]$, which is varied from $[1,10]$ to $[20,30]$. We generate $|F|$ fresh tokens, where $|F|$ is varied from 0 to 20 . In Monero, as shown in the real data sets, the number of tokens in an hour is less than 800. Thus, this setting is realistic and can cover most application scenarios. Moreover, we set the HT of each token by the normal distribution with variances from 8 to 16. When the variance is 16 and the number of tokens is around


Figure 5: Effect of $c_{\tau}$ of the Recursive ( $c_{\tau}, \ell_{\tau}$ )-Diversity (Real).

(a) Size

(b) Running Time

Figure 6: Effect of $c_{\tau}$ of the Recursive ( $c_{\tau}, \ell_{\tau}$ )-Diversity (Real).
800, the number of tokens from the same historical transaction is around 16. In Monero [2], the maximal number of tokens from the same historical transaction is 16 . Thus, our setting is practical.
Compared Approaches. Let $T M_{-} P$ and $T M_{-} G$ indicate the TokenMagic framework with the Progressive Algorithm and the Gametheoretic Algorithm to generate RSs, respectively. We conduct experiments on both the real data sets and the synthetic data sets to evaluate the effectiveness and efficiency of our two approaches, $T M_{-} P$ and $T M_{-} G$, in terms of the size of the new RS and the running time. We compare our approaches with two baseline algorithms, noted as $T M_{-} S$ and $T M_{-} R$. The $T M_{-} S$ denotes the TokenMagic framework using the Smallest Algorithm to generate RSs, which repeatedly adds the super RS or the fresh token with smallest size to the RS until the new RS is eligible. The $T M_{-} R$ is the TokenMagic framework with the Random Algorithm to generate RSs, which repeatedly adds a super RS or a fresh token in random until the new RS is eligible.

Table 2 and Table 3 show experiment settings on two data sets, where the default values of parameters are in bold font. In each set of experiments, we vary one parameter, while keeping other parameters to their default values. For each experiment, we sample 1000 problem instances. We report the average value of the running time and the size of the RS. All experiments were run on an Intel CPU @ 2.2 GHz with 16GM RAM in Java.

### 7.2 Result of the Small-Scale Data Sets

We first run the TokenMagic framework with BFS to generate RSs (noted as $T M_{\_} B$ ) on a synthetic small-scale data set. There are 20 tokens in $T$. Each RS requires a recursive (5,3)-diversity. Figure 4 illustrates the running time of the generation of the $i^{\text {th }} \mathrm{RS}$. As illustrated in Section 5, the running time of $T M_{-} B$ increases exponentially. The running time of generating $8^{t h} \mathrm{RS}$ is around 2 hours. Thus, it is important to implement the practical configurations in Subsection 6.1 and use our practical algorithms, the Progressive Algorithm and the Game-theoretic Algorithm, to generate RSs.


Figure 7: Effect of the variance of tokens' distribution (Synthetic).

### 7.3 Results on Real Data Sets

To exam the performance of $T M_{-} P$ and $T M_{-} G$, we conduct experiments on the real data sets.
Effect of $c_{\tau}$ for the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity requirement. Figure 5 illustrates the experimental result on different $c_{\tau}$, from 0.2 to 1 . In Figure 5(a), when $c_{\tau}$ gets larger, the sizes of the new RSs that generated by four approaches decrease. Because when $c_{\tau}$ is larger, it is easier to satisfy the diversity constraint and the new RS can contain fewer tokens from infrequent HTs. The sizes of the new RSs that generated by TM_G and TM_P are much less than the sizes of the new RSs that generated by two baseline algorithms. In Figure 5(b), at beginning, when $c_{\tau}$ increases, the running time of four approaches decreases. Because when $c_{\tau}$ is larger, it is easier to satisfy the diversity constraint. Then, when $c_{\tau}$ is large enough, the running time of four approaches keeps stable. Because when $c_{\tau}$ is large enough, the constraint caused by $c_{\tau}$ is relaxed and the running time is dominated by other factors.
Effect of $\ell_{\tau}$ for the recursive $\left(c_{\tau}, \ell_{\tau}\right)$-diversity requirement. Figure 6 illustrates the experimental result on different $\ell_{\tau}$, from 20 to 60. In Figure 6(a), when $\ell_{\tau}$ increases, the sizes of the new RSs increase linearly as proved in Theorem 6.5 and 6.7. Because when $\ell_{\tau}$ is larger, the new RS has to contain more tokens from infrequent HTs to meet the recursive ( $c_{\tau}, \ell_{\tau}$ ) -diversity constraint. In Figure 6(b), when $\ell_{\tau}$ increases, the running time of four approaches increases. Among four approaches, $T M_{-} G$ is the slowest and it is more sensitive to the change of $\ell_{\tau}$ compared with the other three algorithms.

In the experiments on the real dataset, we find that the sizes of the new RSs generated by our proposed two approaches are much less than the sizes of the new RSs that generated by two baseline algorithms. However, since the size of each super RS is the same and the distribution of HTs of tokens is almost uniform, the difference between the performance of four approaches is not obvious.

### 7.4 Results on Synthetic Data Sets

We further examine the effects of the distribution of HTs of tokens, the number of super RSs, the number of fresh tokens, and the size of each super RS on the synthetic data sets.
Effect of the variance $\sigma$ of distribution of the HTs of tokens. Figure 7 illustrates the experimental results on different $\sigma$, from 8 to 16. In Figure 7(a), when $\sigma$ gets larger, the sizes of the new RSs that generated by four approaches decrease. When $\sigma$ gets larger, tokens in $T$ would come from more different HTs and the number of times a HT outputting a token in $T$ decrease. Thus, when $\sigma$ gets larger, it is easier to satisfy the diversity constraint and the new RS can contain fewer tokens from infrequent HTs. In Figure 7(b), when $\sigma$

(a) Size

(b) Running Time

Figure 8: Effect of the number of super RSs $|S|$ (Synthetic).
increases, the running time of four approaches decreases. Although $T M_{-} P$ also takes more time than the two baseline algorithms, the time it takes is much less than that of $T M_{-} G$.
Effect of the size of each super RS $\left|s_{i}\right|$. Figure 9 shows the experimental result on different ranges of $\left|s_{i}\right|$, from $[1,10]$ to [20, 30]. In Figure 9(a), when $\left|s_{i}\right|$ increases, the sizes of the new RSs generated by four approaches increase. Because of the first practical configuration, each super RS cannot be partially picked in the new RS. Thus, when $\left|s_{i}\right|$ increases, the sizes of new RSs increase. In Figure 8(b), when $\left|s_{i}\right|$ increases, the running time of four approaches increases. Because when $\left|s_{i}\right|$ increases, the number of tokens $|T|$ increases.
Effect of the number of super RSs $|S|$. Figure 8 illustrates the experimental result on different numbers of super RSs, $|S|$, from 10 to 90 . In Figure 8(a), when $|S|$ gets larger, the sizes of the new RS generated by $T M_{-} R$ keep stable but the sizes of the new RSs generated by the other three approaches decrease. Since $T M_{-} R$ picks mixins randomly, the increment of the number of super RSs has no effect on the size of the new RS. But when $|S|$ gets larger, there are more candidate super RSs can be picked and the other three algorithms can find RSs with smaller sizes. As shown in Figure 8(b), when $|S|$ increases, the running time of four approaches increases. Specifically, the running time of $T M_{-} P$ increases quadratically and the running time of $T M_{-} G$ increases cubically, which confirm the time complexities we analyzed in Section 6.
Effect of the number of fresh tokens $|F|$. Figure 10 illustrates the experimental result on different numbers of fresh tokens, $|F|$, from 0 to 20. In Figure 10(a), when $|F|$ gets larger, the sizes of the new RS generated by $T M_{-} R$ keep stable but the sizes of the new RSs generated by the other three approaches decrease. Since $T M_{-} R$ picks mixins randomly, the increment of $|F|$ has no effect on the size of the new RS. But when $|F|$ gets larger, there are more candidate fresh tokens can be picked and the other three algorithms can find RSs with smaller sizes. In Figure 10(b), when $|F|$ increases, the running time of four approaches also increases. Because when $|F|$ is larger, there are more candidate fresh tokens that can be picked, which increases the time complexity of each algorithm.
Summary of Results. For blockchain systems adopting RS schemes to preserve users' privacy, our methods can help to resist the "chainreaction" attack. The TM_G outputs the RS with the smallest size while it is the slowest. Since RSs are generated offline and the generation time of a RS does not affect the throughput of a blockchain system, it does not matter that the running time of TM_G is a little higher, compared with other approaches. Thus, for the applications where the transaction fee (proportional to the size of RS) is high, like cryptocurrencies (e.g., Moner [2], Bytecoin[31]) and storage


Figure 9: Effect of the size of each super RS $\left|s_{i}\right|$ (Synthetic).
sharing systems (e.g., Oodrive [32]), users can save transaction fee from using TM_G, since the size of a RS generated by TM_G is the smallest. For the applications where users need to make transactions fast, like healthcare systems (e.g., MedBlock [33]) and e-voting systems (e.g., Blockvotes [34]), TM_G is infeasible. For example, in a polling station, if the time cost of generating a RS for a vote increases by 100 ms , it would delay more than one minute for a queue with 1000 voters. Thus, for the applications where users should make transactions quickly and the size of a RS is not very sensitive, like healthcare systems (e.g., MedBlock [33]) and e-voting systems (e.g., Blockvotes [34]), TM_P is suitable.

## 8 RELATED WORK

To solve the privacy problem, some researchers have proposed some privacy-preserved blockchain systems. These works can be classified into two categories. The works in the first category focus on developing mixing protocols. In these protocols, anonymous service providers use mixing protocols to confuse the trails of transactions. However, there is a risk of theft by the service. Researchers proposed a novel protocol, Mixcoin [35], with an accountability mechanism, to avoid such risk. Generally, the server would sign warranties to users and if the server steals users' tokens, users can publish warranties to expose the theft. However, by these mixing protocols, a user has to communicate with others, and she/he cannot make a transaction by herself/himself. When the communication channel is unsafe, users' privacy may be disclosed.

The second category focus on developing advanced encryption methods. The ZCash privacy-preserving blockchain system [36] is built on the zero-knowledge succinct non-interactive argument of knowledge (zk-SNARKs) protocol [37]. However, it needs a trusted entity to initially set up the system, which is unacceptable in many distributed applications. Since the RS schemes does not require an initial set up by a trusted entity, they are widely implemented in blockchain systems in various domains, like cryptocurrency [2, 31], e-voting [3, 34], healthcare [4] and storage sharing [5, 32]. The RS schemes are used to conceal the real token of a transaction and a token can represent the identity of a user, not only cryptocurrency. In a cryptocurrency system [2], the RS scheme is adopted to hide the real token of an input in a transaction and in a RS, a mixin is a UTXO. In an e-voting system [3], the RS scheme is adopted to hide the real user of a vote, and in a RS, a mixin is a user's identity. In a healthcare system [4], the RS scheme is adopted to hide the real patient of a transaction, and in a RS, a mixin is a patient's identity. In a storage sharing system [5], the RS scheme is adopted to hide the real user of a transaction who wants to operate the data, and in a RS, a mixin is a user's identity. These works are the variants of


Figure 10: Effect of the number of fresh tokens $|F|$ (Synthetic).
the RS scheme adopted in the Monero system, and they all do not consider the anonymity of a RS.

As introduced in Subsection 2.1, the generation of a RS includes two processes (i.e., Step 1 and Step 2). Step 1 picks some mixins and Step 2 runs an encryption algorithm to generate a RS using the selected mixins. The existing works focus on decreasing the number of auxiliary parameters in Step 2. For the advanced RS scheme [16], the number of auxiliary parameters is $O(\log n)$ where $n$ is the number of tokens. However, it randomly pick mixins in Step 1 [2], whose time complexity is $O(n)$ where $n$ is the number of tokens. Our methods focus on Step 1. The time complexities of the Progressive Algorithm and the Game-theoretic Algorithm are $O\left(n^{2}\right)$ and $O\left(n^{3}\right)$, respectively. Although the time cost of our algorithms is higher, the RSs generated by our algorithms can resist the "chain-reaction" analysis and the homogeneity attack. Besides, since Step 1 is made offline, the increase of its time complexity does not affect the throughput of a system. For Step 2, we use the advanced encryption algorithm [16].

## 9 CONCLUSION

In this paper, we target on solving privacy weakness in block-chain systems that adopt RS schemes. We formulate the diversity-aware mixin selection problem, which aims to find a RS that satisfies the diversity requirement and contains a minimal number of tokens. We prove the problem is \#P. We propose an exact algorithm to solve it when the input is small. Besides, we propose two practical configurations with two approximation algorithms, the Progressive Algorithm and the Game-theoretic Algorithm, with theoretic guarantees. When evaluated on the real and synthetic data sets, our approaches achieved clearly better performance than two baselines.

## 10 ACKNOWLEDGMENT

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